Effective Descriptive Set Theory

what it is about

Lecture 3, Structure theory for pointclasses

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Outline

Lecture 1. Recursion in Polish spaces

Lecture 2. Effective Borel, analytic and co-analytic pointsets

Lecture 3. Structure theory for pointclasses

★ *Constructively defined sets and functions have good properties*
e.g., *every uncountable $\mathbf{\Sigma}^1_1$ pointset has a non-empty perfect subset*

- We have reduced this to showing that

$$\boxed{\{y \in \mathcal{Y} : y \in \Delta^1_1[x]\} \text{ is } \Pi^1_1[x]}, \text{ where}$$

$$y \in \Delta^1_1[x] \iff \mathcal{U}(y) = \{s : y \in N_s\} \text{ is } \Delta^1_1[x] \iff \{y\} \text{ is } \Sigma^1_1[x]\}$$

Def $\boxed{y \leq^{\mathrm{HYP}} x \iff y \in \Delta^1_1[x] \quad (y \in \mathcal{Y}, x \in \mathcal{X})}$

- Hyperarithmetical reducibility, much studied when $\mathcal{Y} = \mathcal{X} = \mathcal{N}$
- We will prove $\boxed{\{(x, y) : y \leq^{\mathrm{HYP}} x\} \text{ is } \Pi^1_1}$, a structure property of Π^1_1

★ *Constructively defined pointclasses have a good structure theory*

⋆ The prewellordering property

Def A (regular) norm on a pointset $P \subseteq \mathcal{X}$ is any mapping

$$\sigma : P \twoheadrightarrow \lambda_\sigma \in \text{Ords};$$

and it is a Γ-norm if the relations

$$x \leq_\sigma^* y \iff x \in P \ \& \ \neg[y \in P \ \& \ \sigma(y) < \sigma(x)],$$
$$x <_\sigma^* y \iff x \in P \ \& \ \neg[y \in P \ \& \ \sigma(y) \leq \sigma(x)]$$

are both in Γ

Def A pointclass Γ is normed if every $P \in \Gamma$ admits a Γ-norm

• This specific definition of a Γ-norm was not formulated until the early 60's, but many ordinal-valued "index functions" on $\mathbf{\Pi}_1^1$ and $\mathbf{\Sigma}_2^1$ pointsets had been studied in the classical theory (especially by Novikov). This definition has the following very useful property:

⋆ $$\boxed{y \notin P \implies \left(x \leq_\sigma^* y \iff x <_\sigma^* y \iff x \in P\right)}$$

⋆ The prewellordering property for Π^1_1

Theorem (PWO(Π^1_1)) Π^1_1 *is normed*

Proof for $\mathcal{X} = \mathcal{N}$, then use the Refined Surjection Theorem.
If $P \in \Pi^1_1(\mathcal{N})$, then there is a recursive $R \subseteq \mathbb{N}^2$ such that

$$\begin{aligned} P(\alpha) &\iff (\forall \beta)(\exists t) R(\overline{\alpha}(t), \overline{\beta}(t)) \\ &\iff \text{the tree } T(\alpha) \text{ on } \mathbb{N} \text{ is well founded} \end{aligned}$$

where $T(\alpha) = \{(\beta(0), \ldots, \beta(i-1)) : (\forall t < i) \neg R(\overline{\alpha}(t), \overline{\beta}(t))\}$

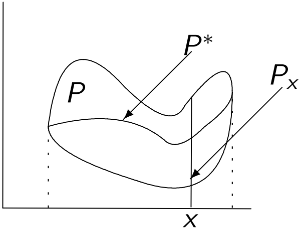
Set $\boxed{\sigma(\alpha) = \text{ the rank of } T(\alpha) \quad (\alpha \in P)}$ and use properties of ranks

Theorem (Norm-Boundedness for Π^1_1) *For any* Π^1_1*-norm* $\sigma : P \twoheadrightarrow \lambda_\sigma$ *on a pointset* $P \subseteq \mathcal{X}$,

$$P \in \mathbf{\Delta}^1_1 \iff \lambda_\sigma < \aleph_1$$

- A useful tool for proving that specific pointsets are not Borel, e.g., $\mathrm{WO} = \{\alpha \in \mathcal{N} : \{(s,t) : \alpha(\langle s, t \rangle) = 1\} \text{ is a wellordering}\}$

Uniformization



Def Suppose $P, P^* \subseteq \mathcal{X} \times \mathcal{Y}$; P^* uniformizes P if

$$P^* \subseteq P \;\&\; (\forall x)[(\exists y)P(x,y) \implies (\exists! y)P^*(x,y)]$$

Theorem (Novikov, Kondo 1938, Addison) *Every* $P \subseteq (\mathcal{X} \times \mathcal{Y})$ *in* Π^1_1 *is uniformized by some* P^* *in* Π^1_1 Deep, central result

★ 1938: Kondo's Theorem and Gödel's construction of L

(s) The Kreisel Uniformization Theorem *Every* $P \subseteq (\mathcal{X} \times \mathbb{N})$ *in* Π^1_1 *is uniformized by some* P^* *in* Π^1_1 Easy but useful

Proof. Let $\sigma : P \to$ Ordinals be a Π^1_1-norm and put

$$P^*(x,t) \iff (\forall s)[(x,t) \leq^*_\sigma (x,s) \;\&\; [(x,t) <^*_\sigma (x,s) \vee t \leq s]]$$

"Soft", axiomatic proofs of structure theorems

• Results marked with **(s)** are proved using only the following properties of Π^1_1:

(a) Π^1_1 contains Σ^0_1 and is closed under recursive substitutions, $\&, \vee, \exists^{\mathbb{N}}, \forall^{\mathbb{N}}$ and $\forall^{\mathcal{Y}}$, for every $\mathcal{Y}$

(b) Π^1_1 is parametrized

(c) $\boxed{\Pi^1_1 \text{ is normed}}$

and so suitable versions of them hold for a large variety of pointclasses, *including the inductive pointsets, the pointsets which are Kleene-semirecursive in* $\exists^{\mathcal{N}}$ *and (under determinacy hypotheses) every* Π^1_{2k+1}

• These "soft" proofs were discovered by work in Kleene's theory of recursion in higher types, the theory of inductive definability and the derivation of consequences of projective determinacy (Spector, Gandy, Kreisel, ynm, Martin, Louveau, Kechris, Harrington, Steel, ...)

The Coding Theorem for $\{y \in \mathcal{Y} : y \in \Delta^1_1[x]\}$

(s) Theorem (after Kleene) *For any $\mathcal{X}, \mathcal{Y}$, there is a partial function $\mathbf{d} : \mathbb{N} \times \mathcal{X} \rightharpoonup \mathcal{Y}$ such that*

(1) $y \in \Delta^1_1[x] \iff y \leq^{\mathrm{HYP}} x \iff (\exists i)[\mathbf{d}(i,x)\downarrow \ \&\ \mathbf{d}(i,x) = y]$

(2) *The following pointsets are Π^1_1:*

$$\{(i,x) : \mathbf{d}(i,x)\downarrow\},$$
$$\{(i,x,y) : \mathbf{d}(i,x)\downarrow \ \&\ \mathbf{d}(i,x) = y\},$$
$$\{(i,x,y) : \mathbf{d}(i,x)\downarrow \ \&\ \mathbf{d}(i,x) \neq y\}$$

Proof outline for $\mathcal{Y} = \mathcal{N}$, then use the Refined Extension Theorem.

Let $\varphi_i : \mathbb{N} \rightharpoonup \mathbb{N}$ be the Turing computable partial function with code i, fix a parametrization G of $\Pi^1_1(\mathcal{X} \times \mathbb{N} \times \mathbb{N})$ and put

$$P(i,x,s,t) \iff \varphi_i \text{ is total } \ \&\ (\forall s)(\exists t)G(\varphi_i,(x,s,t))$$

Fix $P^* \subseteq P$ so that $\boxed{(\exists t)P(i,x,s,t) \implies (\exists! t)P^*(i,x,s,t)}$ and set

$$\mathbf{d}(i,x) = \alpha \iff (\forall s)P^*(i,x,s,\alpha(s))$$

The Effective Perfect Set Theorem, concluded

(s) Theorem (after Kleene) *For any $\mathcal{X}, \mathcal{Y}$, there is a partial function $\mathbf{d} : \mathbb{N} \times \mathcal{X} \rightharpoonup \mathcal{Y}$ such that*

(1) $y \in \Delta^1_1[x] \iff y \leq^{\mathrm{HYP}} x \iff (\exists i)[\mathbf{d}(i,x)\!\downarrow\ \&\ \mathbf{d}(i,x) = y]$

(2) *The following pointsets are Π^1_1:*

$$\{(i,x) : \mathbf{d}(i,x)\!\downarrow\},$$
$$\{(i,x,y) : \mathbf{d}(i,x)\!\downarrow\ \&\ \mathbf{d}(i,x) = y\},$$
$$\{(i,x,y) : \mathbf{d}(i,x)\!\downarrow\ \&\ \mathbf{d}(i,x) \neq y\}$$

$\Rightarrow \{(x,y) : y \leq^{\mathrm{HYP}} x\}$ *is* Π^1_1

This completes the proof of the

Effective Perfect Set Theorem *For $A \in \Sigma^1_1[x](\mathcal{Y})$,*

A has a non-empty perfect subset

$\iff$ *A has a member which is not $\Delta^1_1[x]$*

Restricted Quantification and Spector-Gandy theorems

(s) Theorem (after Kleene) *If* $Q \in \Pi^1_1(\mathcal{X} \times \mathcal{Y})$ *and*

$$P(x) \iff (\exists y \leq^{\mathsf{HYP}} x) Q(x, y),$$

then P *is also* Π^1_1

Proof. $P(x) \iff (\exists i)\Big(\mathbf{d}(i,x)\downarrow \ \& \ (\forall y)[\mathbf{d}(i,x) \neq y \vee Q(x,y)]\Big)$

Theorem (Spector-Gandy) *Every* $P \in \Pi^1_1(\mathbb{N})$ *satisfies an equivalence*

$$P(i) \iff (\exists \alpha \in \mathsf{HYP}) Q(i, \alpha)$$

with some $Q \in \Pi^0_1(\mathbb{N} \times \mathcal{N})$*; more generally, if* $P \in \Pi^1_1(\mathcal{X})$*, then*

$$P(x) \iff (\exists \alpha \leq^{\mathsf{HYP}} x) Q(x, \alpha)$$

with some $Q \in \Pi^0_1(\mathcal{X} \times \mathcal{N})$

- There are several proofs of the Spector-Gandy Theorem, none of them simple—it is certainly one of the jewels of the effective theory

$\star$ Δ^1_1 functions and Lusin's characterization of **B**

Def (Δ^1_1 functions) A (total) function $f : \mathcal{X} \to \mathcal{Y}$ is effectively Borel measurable or Δ^1_1 if its graph $\{(x, y) : f(x) = y\}$ is Δ^1_1

(s) Theorem *If $A \subseteq \mathcal{X}$ is Δ^1_1, $f : \mathcal{X} \to \mathcal{Y}$ is Δ^1_1 and f is injective on A, then $f[A]$ is Δ^1_1*

Proof. $y \in f[A] \iff (\exists x)[x \in A \;\&\; f(x) = y]$ (so $f[A]$ is Σ^1_1)

$\iff (\exists! x)[x \in A \;\&\; f(x) = y] \iff (\exists x \leq^{\mathrm{HYP}} y)[x \in A \;\&\; f(x) = y]$

and so $f[A]$ is also Π^1_1, by the Restricted Quantification Theorem

Theorem (Effective version) *A set $B \subseteq \mathcal{X}$ is Δ^1_1 if and only if $B = f[A]$ for some Π^0_1 set $A \subseteq \mathcal{N}$ and a recursive $f : \mathcal{N} \to \mathcal{X}$ which is injective on A*

Theorem (Classical version, Lusin 1917) *A set $B \subseteq \mathcal{X}$ is Borel if and only if $B = f[A]$ for some closed $A \subseteq \mathcal{N}$ and a continuous $f : \mathcal{N} \to \mathcal{X}$ which is injective on A*

Δ^1_1 isomorphisms

Theorem (Classical) *Every uncountable Polish space is Borel isomorphic with the Baire space $\mathcal{N}$*

Theorem (Effective) *Every perfect recursive Polish space is Δ^1_1 isomorphic with $\mathcal{N}$*

Theorem *Every uncountable recursive Polish space $\mathcal{X}$ is $\Delta^1_1[\mathbf{p}(\mathcal{X})]$ isomorphic with $\mathcal{N}$*, where $\mathbf{p}(\mathcal{X})$ is the characteristic function of

$$P_{\mathcal{X}}(s) \iff N(\mathcal{X}, s) \text{ is uncountable}$$

computed relative to a compatible pair $(d, \mathbf{r})$ of $\mathcal{X}$

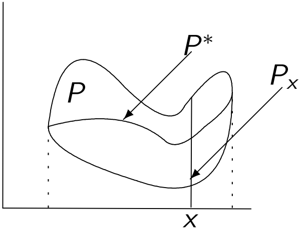
- $P_{\mathcal{X}}$ is Σ^1_1 but not (in general) Δ^1_1

Theorem (Gregoriades) *There exist uncountable recursive Polish spaces which are not Δ^1_1 isomorphic with $\mathcal{N}$*

- Gregoriades has initiated a deep study of the reducibility relation

$$\mathcal{X} \leq^{\mathrm{HYP}} \mathcal{Y} \iff \textit{there exists a } \Delta^1_1 \textit{ embedding of } \mathcal{X} \textit{ into } \mathcal{Y}$$

⋆ The Δ-Uniformization Criterion



(s) Theorem *For every $P \in \Delta^1_1[\varepsilon](\mathcal{X} \times \mathcal{Y})$, the following are equivalent*

(1) *Some $P^* \in \Delta^1_1[\varepsilon](\mathcal{X} \times \mathcal{Y})$ uniformizes P*

(2) *For every $x \in \mathcal{X}$, $(\exists y)P(x,y) \implies (\exists y \leq^{\mathrm{HYP}} (\varepsilon, x))P(x,y)$*

Moreover, if (1) *or* (2) *holds, then* $\mathrm{proj}(P) = \{x : (\exists y)P(x,y)\}$ *is* $\Delta^1_1[\varepsilon]$

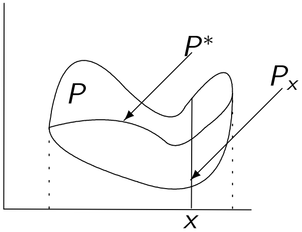
Proof. (1) $\implies$ (2): If $P^*(x,y)$, then $\{y\} \in \Delta^1_1[\varepsilon, x]$, so $y \in \Delta^1_1[\varepsilon, x]$

(2) $\implies$ (1): Set $Q(x,i) \iff [\mathbf{d}(i,x)\!\downarrow\ \&\ P(x, \mathbf{d}(i,x))]$, use Kreisel Uniformization to get Q^* and use $\mathbf{d}$ again to get P^* from Q^*

The second claim follows by the Restricted Quantification Theorem

$\boxed{\star\ \text{Characteristic result of EDST}}$ *Is there a classical version of it?*

Borel sets with countable sections



Theorem (classical, Lusin 1930) *If every section P_x of a Borel set $P \subseteq \mathcal{X} \times \mathcal{Y}$ is countable, then* proj(P) *is Borel and P can be uniformized by a Borel set set P^**

(s) Theorem (effective) *If every section P_x of a $\Delta^1_1[\varepsilon]$ set $P \subseteq \mathcal{X} \times \mathcal{Y}$ is countable, then* proj(P) *is $\Delta^1_1[\varepsilon]$ and P can be uniformized by a $\Delta^1_1[\varepsilon]$ set P^**

Proof. Every P_x is $\Delta^1_1[\varepsilon, x]$, so if it is countable it is contained in $\{y : y \in \Delta^1_1[\varepsilon, x]\}$ by the Effective Perfect Set Theorem; and so the Δ-Uniformization Criterion applies

Monotone inductive definitions

Def An operator $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ on the powerset of a set X is monotone if $\boxed{S \subseteq T \implies \Phi(S) \subseteq \Phi(T) \quad (S, T \subseteq X)}$

$\Rightarrow$ *Every monotone* $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ *has a least fixed point* $\overline{\Phi}$ *characterized by*

$$\Phi(\overline{\Phi}) = \overline{\Phi}, \quad (\forall S \subseteq X)[\Phi(S) \subseteq S \implies \overline{\Phi} \subseteq S]$$

$\Rightarrow \overline{\Phi} = \bigcap\{S \subseteq X : \Phi(S) \subseteq S\} = \bigcup_\xi \overline{\Phi}_\xi,$

where by ordinal recursion, $\overline{\Phi}_\xi = \Phi(\bigcup_{\eta<\xi} \overline{\Phi}_\eta)$

- For example, the set K of Borel codes is the least fixed point $\overline{\Phi}^b$ of

$$\Phi^b(S) = \{\alpha : \alpha(0) = 0 \vee [\alpha(0) \neq 0 \ \&\ (\forall i)[(\alpha)_i \in S]]\} \quad (S \subseteq \mathcal{N})$$

- The next result often gives the best explicit characterization of $\overline{\Phi}$

$\star$ The Normed Induction Theorem

Def A monotone operator $\Phi : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ is $\boxed{\Gamma \textit{ on } \Gamma}$ if

$$Q \in \Gamma(\mathcal{X} \times \mathcal{Y}) \implies \{(x, y) : x \in \Phi(\{x' : Q(x', y)\} \in \Gamma$$

(s) Theorem *If* $\Phi : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ *is* Π^1_1 *on* Π^1_1, *then* $\overline{\Phi}$ *is* Π^1_1

$\Rightarrow$ K *is* Π^1_1 (which, however, has an elementary proof)

Theorem (ynm, 1974) *Let* Γ *be a pointclass and* $\mathcal{X}$ *a space. If*

(1) Γ *is parametrized,*
(2) *some parametrization* G *of* $\Gamma(\mathcal{X})$ *admits a* Γ*-norm, and*
(3) $\Phi : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ *is* Γ *on* Γ,

then the least fixed point $\overline{\Phi} \subseteq \mathcal{X}$ *is in* Γ

- The hypotheses hold for Σ^0_{k+1}, Π^1_1 and any $\mathcal{X}$, and for Σ^0_1 and $\mathbb{N}^n, \mathcal{N}^n$

- Debs 2008 uses this result (and many other things) to obtain some interesting applications to *Rosenthal compacta* which do not (as yet) have classical proofs

Proof of the Normed Induction Theorem

Theorem *Let Γ be a pointclass and $\mathcal{X}$ a space. If*

(1) *Γ is parametrized,*
(2) *some parametrization G of $\Gamma(\mathcal{X})$ admits a Γ-norm, and*
(3) *$\Phi : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ is Γ on Γ,*

then the least fixed point $\overline{\Phi} \subseteq \mathcal{X}$ is in Γ

Proof. Let $\sigma : G \twoheadrightarrow \lambda_\sigma$ be a Γ-norm on the $G \in \Gamma(\mathcal{N} \times \mathcal{X})$ given by (2) and by the 2nd RT choose a recursive $\widetilde{\varepsilon}$ so that

$$G(\widetilde{\varepsilon}, x) \iff x \in \Phi(\{x' : (\widetilde{\varepsilon}, x') <^*_\sigma (\widetilde{\varepsilon}, x)\})$$

Using the monotonicity of Φ, prove that

(a) $G(\widetilde{\varepsilon}, x) \implies x \in \overline{\Phi}$, by induction on $\sigma(\widetilde{\varepsilon}, x)$, and

(b) $x \in \overline{\Phi}_\xi \implies G(\widetilde{\varepsilon}, x)$, by induction on ξ

For (b), assume the ind. hyp, $x \in \overline{\Phi}$ and $\neg G(\widetilde{\varepsilon}, x)$; note that $(\widetilde{\varepsilon}, x') <^*_\sigma (\widetilde{\varepsilon}, x) \iff G(\widetilde{\varepsilon}, x')$, and $\overline{\Phi}_{<\xi} \subseteq G_{\widetilde{\varepsilon}}$ by the ind. hyp., so $x \in \overline{\Phi}_\xi \implies x \in \Phi(\overline{\Phi}_{<\xi}) \implies x \in \Phi(G_{\widetilde{\varepsilon}}) \implies G(\widetilde{\varepsilon}, x)$

Asymmetric open games

- For any $\mathcal{X}$ and $A \subseteq \mathcal{X}^{<\omega} \times \mathbb{N}^{<\omega}$, consider the two-player game

$$G(\mathcal{X}, A) \qquad \begin{array}{c|cccccccc} \text{I} & x_0 & & x_1 & & x_2 & & \cdots & \\ & & & & & & & & \\ \text{II} & & t_0 & & t_1 & & t_2 & & \cdots \end{array}$$

where I plays in $\mathcal{X}$, II plays in $\mathbb{N}$ and

$$\boxed{\text{II wins if for some } n, (\vec{x}, \vec{t}) = ((x_0, \ldots, x_{n-1}), (t_0, \ldots, t_{n-1})) \in A}$$

- $G(\mathcal{X}, A)$ is determined by the Gale-Stewart Theorem

Def $W(\vec{x}, \vec{t}) \iff (x_0, t_0, \ldots, x_{n-1}, t_{n-1})$ is a winning position for II

(s) Theorem *If A is Π^1_1 ($\boldsymbol{\Pi}^1_1$), then*

(1) *W is Π^1_1 ($\boldsymbol{\Pi}^1_1$) and*

(2) *if* II *wins the game, then she has a Δ^1_1 ($\boldsymbol{\Delta}^1_1$) winning strategy* $\sigma : \mathcal{X}^{<\omega} \to \mathbb{N}$

- With $\mathcal{X} = \mathbb{N}$, the effective version is well known
- I don't know a classical proof for the classical version with $\mathcal{X} = \mathcal{N}$

$\star$ More on Γ on Γ

Def A relation $\Phi \subseteq \mathcal{X} \times \mathcal{P}(\mathcal{Z})$ is Γ on Γ if for every $Q \in \Gamma(\mathcal{Y} \times \mathcal{Z})$ the pointset

$$P(x, y) \iff \Phi(x, \{z : Q(y, z)\})$$

is in Γ

Def Similarly, without a parameter, $\Phi \subseteq \mathcal{P}(\mathcal{Z})$ is Γ on Γ if for every $Q \in \Gamma(\mathcal{Y} \times \mathcal{Z})$, the pointset

$$P(y) \iff \Phi(\{z : Q(y, z)\})$$

is in Γ

Monotonicity of Γ on Γ relations

Theorem *If Γ is parametrized and closed under & and $\vee$, then every $\Phi \subseteq \mathcal{X} \times \mathcal{P}(\mathcal{Z})$ which is Γ on Γ is monotone on Γ, i.e.,*

$$\Big(A, B \in \Gamma(\mathcal{Z}) \;\&\; \Phi(x, A) \;\&\; A \subseteq B\Big) \implies \Phi(x, B)$$

Proof. Suppose x, A, B satisfy the hypotheses, fix a parametrization G of $\Gamma(\mathcal{X} \times \mathcal{Z})$ and choose a recursive $\widetilde{\varepsilon}$ $\boxed{\text{by the 2nd RT}}$ such that

$$G(\widetilde{\varepsilon}, x, z) \iff z \in A \vee \Big(\Phi(x, \{z : G(\widetilde{\varepsilon}, x, z)\}) \;\&\; z \in B\Big)$$

Now $\Phi(x, \{z : G(\widetilde{\varepsilon}, x, z)\})$; because if not, then $\{z : G(\widetilde{\varepsilon}, x, z)\}) = A$ and the hypothesis gives $\Phi(x, \{z : G(\widetilde{\varepsilon}, x, z)\})$. So

$$G(\widetilde{\varepsilon}, x, z) \iff z \in A \vee z \in B \iff z \in B$$

and the boxed claim yields the required $\Phi(x, B)$

Π^1_1-reflection

(s) Theorem *If $\Phi \subseteq \mathcal{P}(\mathcal{Z})$ is Π^1_1 on Π^1_1, then for every A,*

$$A \in \Pi^1_1(\mathcal{Z}) \ \&\ \Phi(A) \implies (\exists B \subseteq A)[B \in \Delta^1_1(\mathcal{Z}) \ \&\ \Phi(B)]$$

Proof. Fix a Π^1_1-norm σ on a parametrization G of $\Pi^1_1(\mathcal{Z} \times \mathcal{N})$, fix recursive $\varepsilon_A, \varepsilon_\Phi$ such that for all z, α with constants $r_0^{\mathcal{N}} \in \mathcal{N}, r_0^{\mathcal{Z}} \in \mathcal{Z}$,

$$G(\varepsilon_A, z, \alpha) \iff z \in A, \quad G(\varepsilon_\Phi, z, \alpha) \iff \Phi(\{z' : G(\alpha, z', r_0^{\mathcal{N}})\})$$

and choose a recursive $\widetilde{\varepsilon}$ $\boxed{\text{by the 2nd RT}}$ such that

$$G(\widetilde{\varepsilon}, z, \alpha) \iff (\varepsilon_A, z, r_0^{\mathcal{N}}) <^*_\sigma (\varepsilon_\Phi, r_0^{\mathcal{Z}}, \widetilde{\varepsilon})$$

Now $G(\varepsilon_\Phi, r_0^{\mathcal{Z}}, \widetilde{\varepsilon})$; because if not, then

$$\{z' : G(\widetilde{\varepsilon}, z', r_0^{\mathcal{N}})\} = \{z' : G(\varepsilon_A, z', r_0^{\mathcal{N}})\} = A$$

and so $G(\varepsilon_\Phi, r_0^{\mathcal{Z}}, \widetilde{\varepsilon})$. Hence $\Phi(\{z : G(\widetilde{\varepsilon}, z, r_0^{\mathcal{N}})\}$, and we can take

$$B = \{z : G(\widetilde{\varepsilon}, z, r_0^{\mathcal{N}})\} = \{z : (\varepsilon_A, z, r_0^{\mathcal{N}}) <^*_\sigma (\varepsilon_\Phi, r_0^{\mathcal{Z}}, \widetilde{\varepsilon})\}$$

Kreisel Compactness

(s) Theorem *Suppose $\{B_i\}_{i\in\mathbb{N}}$ is an indexed family of subsets of $\mathcal{X}$ and $I \subseteq \mathbb{N}$ so that*

(1) *the pointset $\{(i,x) : x \in B_i\}$ is Σ^1_1, and*
(2) *I is Π^1_1*

Then

$$(*) \qquad (\forall J \in \Delta^1_1(\mathbb{N}), J \subseteq I)[\bigcap_{i\in J} B_i \neq \emptyset] \implies \bigcap_{i\in I} B_i \neq \emptyset$$

Proof. The contrapositive of $(*)$ is

$$\bigcap_{i\in I} B_i = \emptyset \implies (\exists J \subseteq I)[J \in \Delta^1_1(\mathbb{N}) \ \&\ \bigcap_{i\in J} B_i = \emptyset]$$

and it is an instance of Π^1_1 reflection on

$$\Phi(A) \iff \bigcap_{i\in A} B_i = \emptyset \quad (A \subseteq \mathbb{N})$$

which is Π^1_1 on Π^1_1

The $\Delta^1_1(\mathcal{X})$ Coding Theorem

(s) Theorem (after Kleene) *For every space $\mathcal{X}$, there is a partial function $\mathbf{D} : \mathbb{N} \rightharpoonup \mathcal{P}(\mathcal{X})$ such that*

(1) $B \in \Delta^1_1(\mathcal{X}) \iff (\exists i)[\mathbf{D}(i)\!\downarrow\ \&\ B = \mathbf{D}(i)]$

(2) $\{i \in \mathbb{N} : \mathbf{D}(i)\!\downarrow\}$ *is* Π^1_1 *and so are the pointsets*

$$\{(i,x) : \mathbf{D}(i)\!\downarrow\ \&\ x \in \mathbf{D}(i)\}, \quad \{(i,x) : \mathbf{D}(i)\!\downarrow\ \&\ x \notin \mathbf{D}(i)\}$$

Proof. Let $\pi : \mathcal{N} \twoheadrightarrow \mathcal{X}$ be a recursive surjection
let $\sigma : G \to \text{Ords}$ be a Π^1_1-norm on a parametrization G of $\Pi^1_1(\mathcal{X})$,
for any $B \in \Delta^1_1(\mathcal{X})$ choose a recursive ε_B such that $B = G_{\varepsilon_B}$,
and then $\boxed{\text{by the 2nd RT}}$ choose a recursive $\widetilde{\varepsilon}$ such that

$$\neg G(\widetilde{\varepsilon}, x) \iff (\exists y)[y \in B \ \&\ \neg(\varepsilon_B, y) \leq^*_\sigma (\widetilde{\varepsilon}, x)]$$

Now $G(\widetilde{\varepsilon}, \pi(\widetilde{\varepsilon}))$; because if not, then $y \in B \iff (\varepsilon_B, y) \leq^* (\widetilde{\varepsilon}, \pi(\widetilde{\varepsilon}))$,
and hence $\neg G(\widetilde{\varepsilon}, \pi(\widetilde{\varepsilon})) \iff (\exists y)[y \in B \ \&\ y \notin B]$. So
$B = \{x : (\varepsilon_B, x) \leq^*_\sigma (\widetilde{\varepsilon}, \pi(\widetilde{\varepsilon}))\}$. This codes every $\Delta^1_1(\mathcal{X})$ set B by the pair $(\varepsilon_B, \widetilde{\varepsilon})$ of two recursive Baire points, which suffices

Call-by-name (intensional) and call-by-value (extensional)

(s) The Myhill-Shepherdson Theorem *A relation $\Phi \subseteq \mathcal{P}(\mathbb{N})$ is Σ^0_1 on Σ^0_1 if and only if for some $R \in \Sigma^0_1(\mathbb{N})$ and every $A \in \Sigma^0_1(\mathbb{N})$,*

$$(*) \qquad \Phi(A) \iff (\exists u, n)[\{(u)_i : i < n\} \subseteq A \,\&\, R(u, n)]$$

(s) Theorem *A relation $\Phi \subseteq \mathcal{P}(\mathcal{X})$ is Π^1_1 on Π^1_1 if and only if for some $R \in \Pi^1_1(\mathbb{N})$ and every $A \in \Pi^1_1(\mathcal{X})$,*

$$(**) \qquad \Phi(A) \iff (\exists i)[\mathbf{D}(i)\!\downarrow \,\&\, \mathbf{D}(i) \subseteq A \,\&\, R(i)]$$

- Γ on Γ definitions are call-by-name (intensional)
  —they use a Γ-definition of A to decide $\Phi(A)$
- $(*)$ and $(**)$ are call-by-value (extensional) characterizations
  —they only use membership in A (and Π^1_1 pointsets) to decide $\Phi(A)$

★ In viewing Π^1_1 definability as a generalized recursion theory on recursive Polish spaces, the correct analogies are

$$\boxed{\Pi^1_1 \sim \Sigma^0_1 \text{ and } \Delta^1_1 \sim \text{finite}} \quad (\text{not } \Delta^1_1 \sim \text{recursive})$$